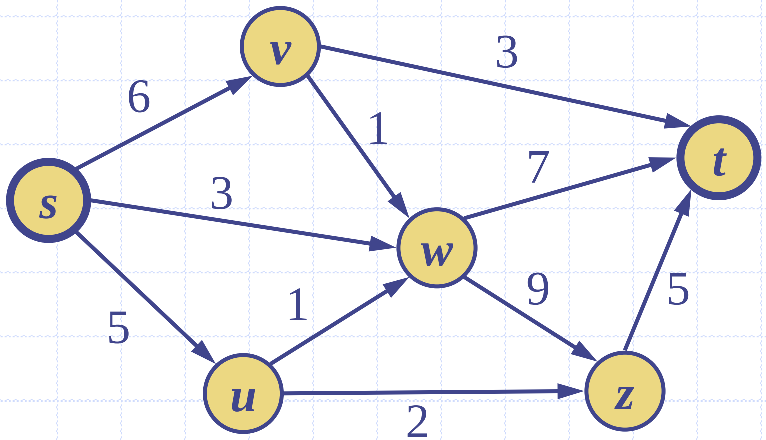
Lecture 14
Maxflow, bipartite matching

CS 161 Design and Analysis of Algorithms

Ioannis Panageas

Flow Network

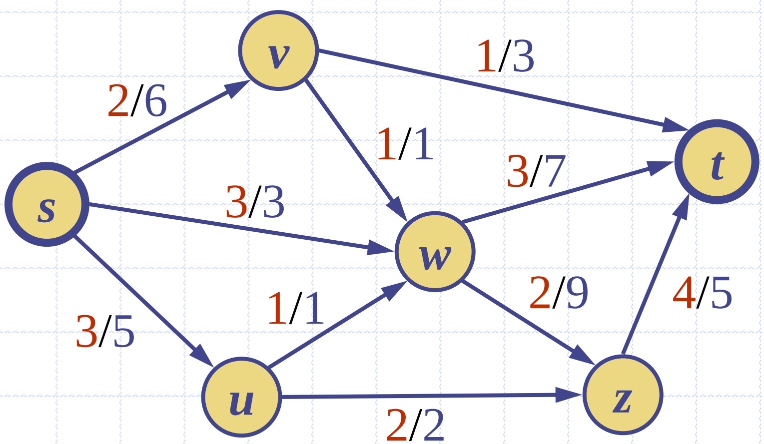
- A flow network (or just network) N consists of
  - A weighted digraph G with nonnegative integer edge weights, where the weight of an edge e is called the capacity $c(e)$ of e
  - Two distinguished vertices, s and t of G, called the source and sink, respectively, such that s has no incoming edges and t has no outgoing edges.
- Example:



Flow

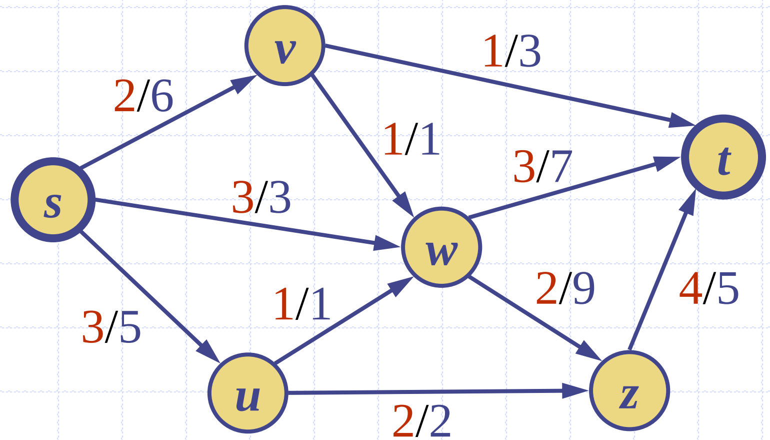
- A flow f for a network N is is an assignment of an integer value $f(e)$ to each edge e that satisfies the following properties:
  - Capacity Rule: For each edge e, $0 \le f(e) \le c(e)$
  - Conservation Rule: For each vertex $v \neq s,t$ $$\sum_{e \in E^-(v)} f(e) = \sum_{e \in E^+(v)} f(e)$$

    where $E^-(v)$ and $E^+(v)$ are the incoming and outgoing edges of v, resp.
- The value of a flow f, denoted $|f|$, is the total flow from the source, which is the same as the total flow into the sink
- Example:

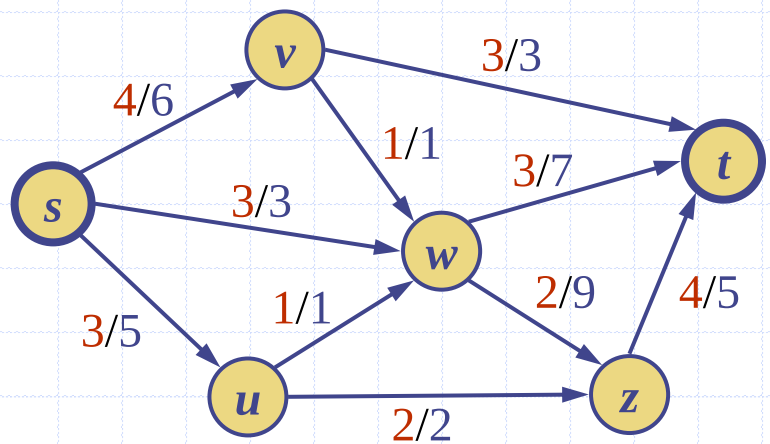


Maximum Flow

- A flow for a network N is said to be maximum if its value is the largest of all flows for N
- The maximum flow problem consists of finding a maximum flow for a given network N
- Applications
  - Hydraulic systems
  - Electrical circuits
  - Traffic movements
  - Freight transportation



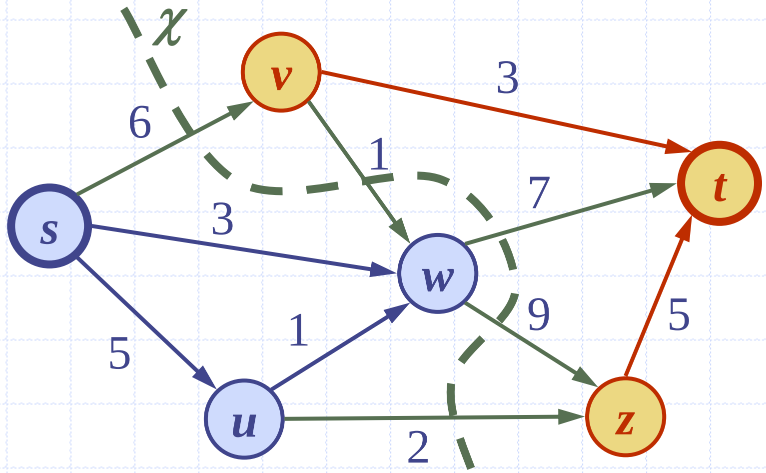
Flow of value 8 = 2 + 3 + 3 = 1 + 3 + 4

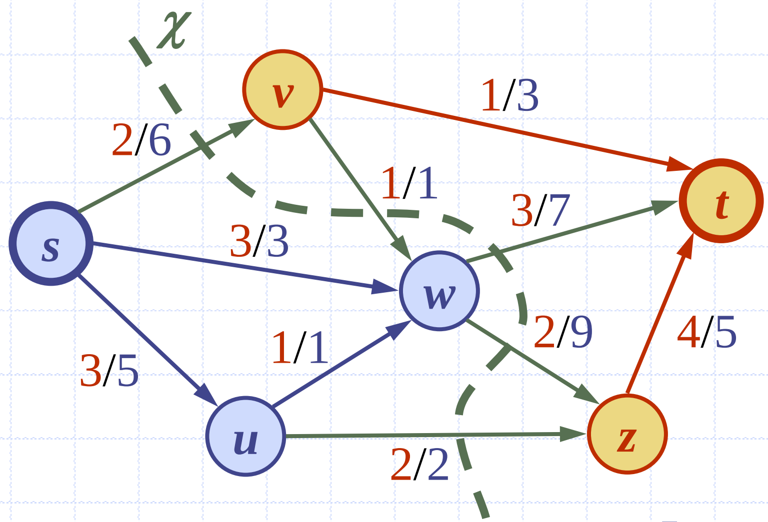


Maximum flow of value 10 = 4 + 3 + 3 = 3 + 3 + 4

Cut

- A cut of a network N with source s and sink t is a partition $\chi = (V_s, V_t)$ of the vertices of N such that $s \in V_s$ and $t \in V_t$
  - Forward edge of cut χ: origin in V_s and destination in V_t
  - Backward edge of cut χ: origin in V_t and destination in V_s
- Flow $f(\chi)$ across a cut χ: total flow of forward edges minus total flow of backward edges
- Capacity $c(\chi)$ of a cut χ: total capacity of forward edges
- Example:
  - $c(\chi) = 24$
  - $f(\chi) = 8$





Flows and Cuts

Lemma:

The flow $f(\chi)$ across any cut χ is equal to the flow value $|f|$

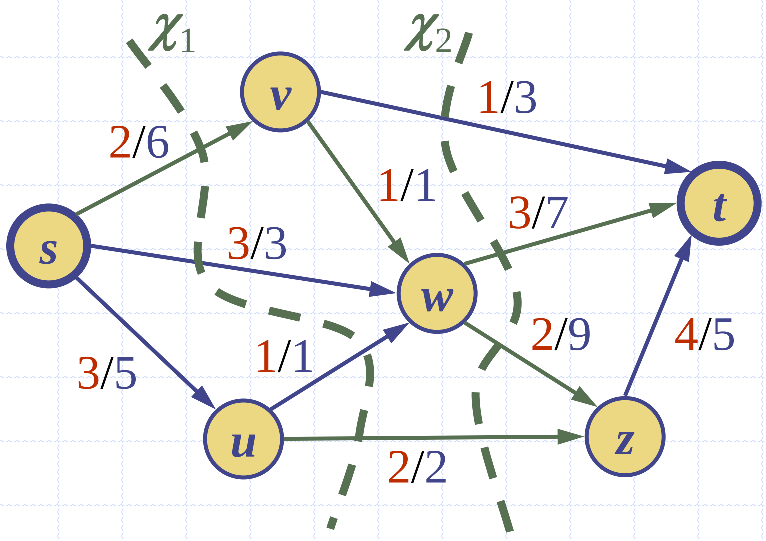
Lemma:

The flow $f(\chi)$ across a cut χ is less than or equal to the capacity $c(\chi)$ of the cut

Theorem:

The value of any flow is less than or equal to the capacity of any cut, i.e., for any flow f and any cut χ, we have

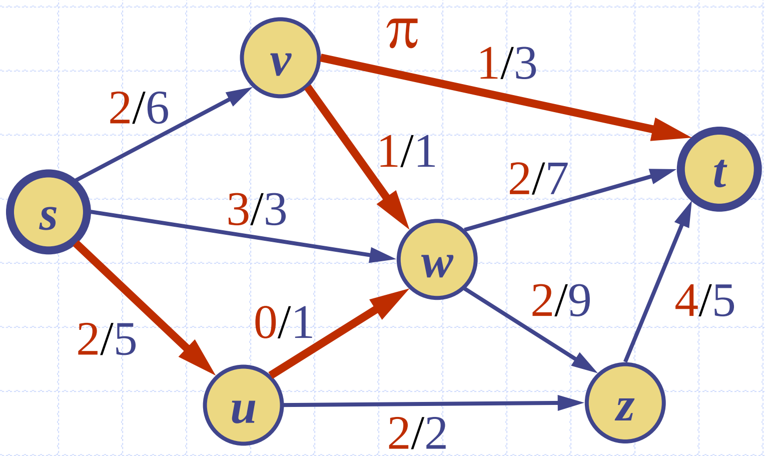
$$|f| \leq c(\chi)$$



$$c(\chi_1) = 12 = 6 + 3 + 1 + 2$$
$$c(\chi_2) = 21 = 3 + 7 + 9 + 2$$
$$|f| = 8$$

Augmenting Path

- Consider a flow f for a network N
- Let e be an edge from u to v:
  - Residual capacity of e from u to v: $\Delta_f(u, v) = c(e) - f(e)$
  - Residual capacity of e from v to u: $\Delta_f(v, u) = f(e)$
- Let π be a path from s to t
  - The residual capacity $\Delta_f(\pi)$ of π is the smallest of the residual capacities of the edges of π in the direction from s to t
- A path π from s to t is an augmenting path if $\Delta_f(\pi) > 0$



$$\Delta_f(s,u) = 3$$
$$\Delta_f(u,w) = 1$$
$$\Delta_f(w,v) = 1$$
$$\Delta_f(v,t) = 2$$
$$\Delta_f(\pi) = 1$$
$$|f| = 7$$

Flow Augmentation

Lemma:

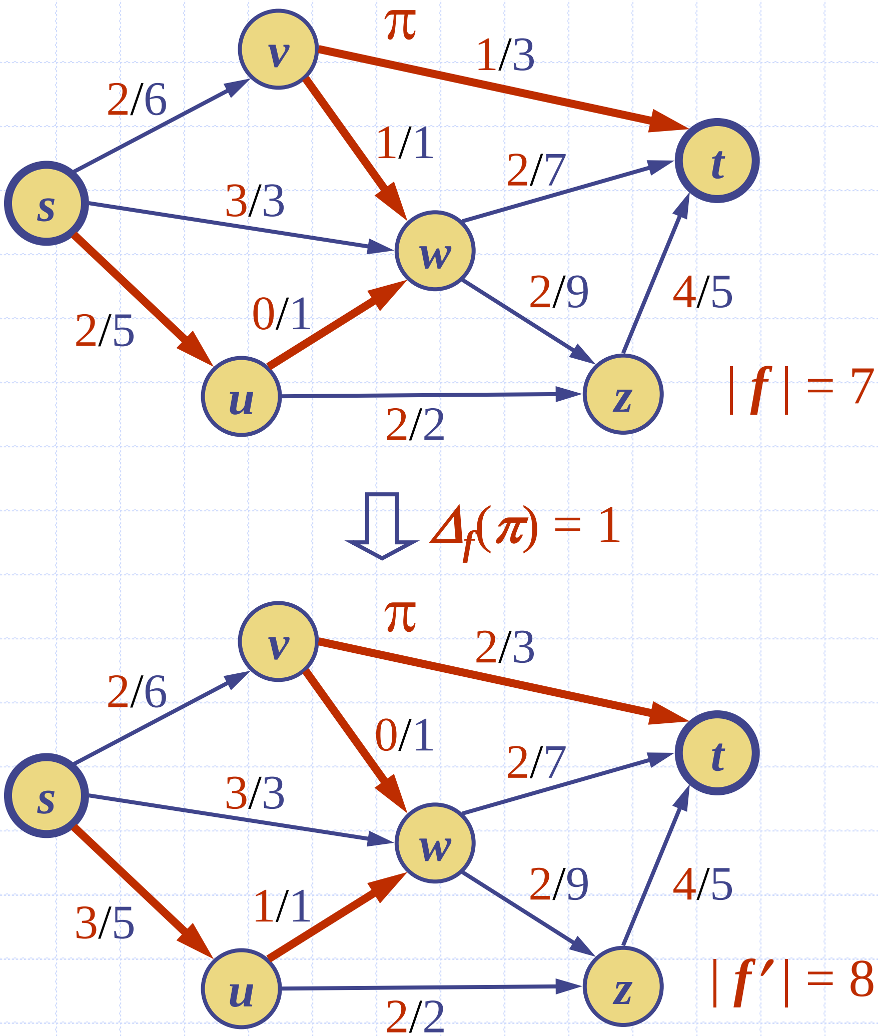
Let π be an augmenting path for flow f in network N. There exists a flow f' for N of value

$$|f'| = |f| + \Delta_f(\pi)$$

Proof:

We compute flow f' by modifying the flow on the edges of π

- Forward edge:
  $$f'(e) = f(e) + \Delta_f(\pi)$$
- Backward edge:
  $$f'(e) = f(e) - \Delta_f(\pi)$$



The Ford-Fulkerson Algorithm

- Initially, $f(e) = 0$ for each edge e
- Repeatedly
  - Search for an augmenting path π
  - Augment by $\Delta_f(\pi)$ the flow along the edges of π
- A specialization of DFS (or BFS) searches for an augmenting path
  - An edge e is traversed from u to v provided $\Delta_f(u, v) > 0$

Algorithm MaxFlowFordFulkerson(N):
    Input: Flow network $N = (G, c, s, t)$
    Output: A maximum flow f for N

```
for each edge e ∈ N do
    f(e) ← 0
stop ← false
repeat
    traverse G starting at s to find an augmenting path for f
    if an augmenting path π exists then
        // Compute the residual capacity Δ_f(π) of π
        Δ ← +∞
        for each edge e ∈ π do
            if Δ_f(e) < Δ then
                Δ ← Δ_f(e)
        for each edge e ∈ π do   // push Δ = Δ_f(π) units along π
            if e is a forward edge then
                f(e) ← f(e) + Δ
            else
                f(e) ← f(e) − Δ        // e is a backward edge
    else
        stop ← true      // f is a maximum flow
until stop
```

Max-Flow and Min-Cut

- Termination of Ford-Fulkerson's algorithm
  - There is no augmenting path from s to t with respect to the current flow f
- Define

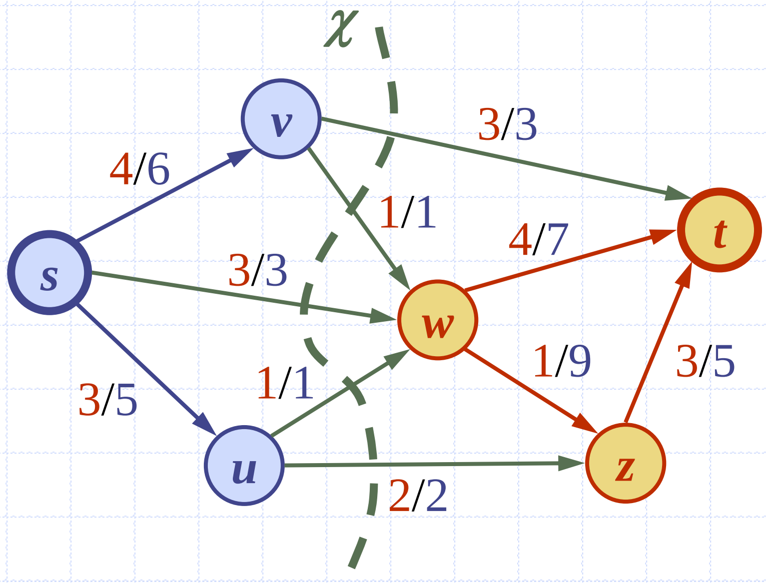
  V_s set of vertices reachable from s by augmenting paths

  V_t set of remaining vertices
- Cut $\chi = (V_s, V_t)$ has capacity

  $c(\chi) = |f|$
  - Forward edge: $f(e) = c(e)$
  - Backward edge: $f(e) = 0$
- Thus, flow f has maximum value and cut χ has minimum capacity

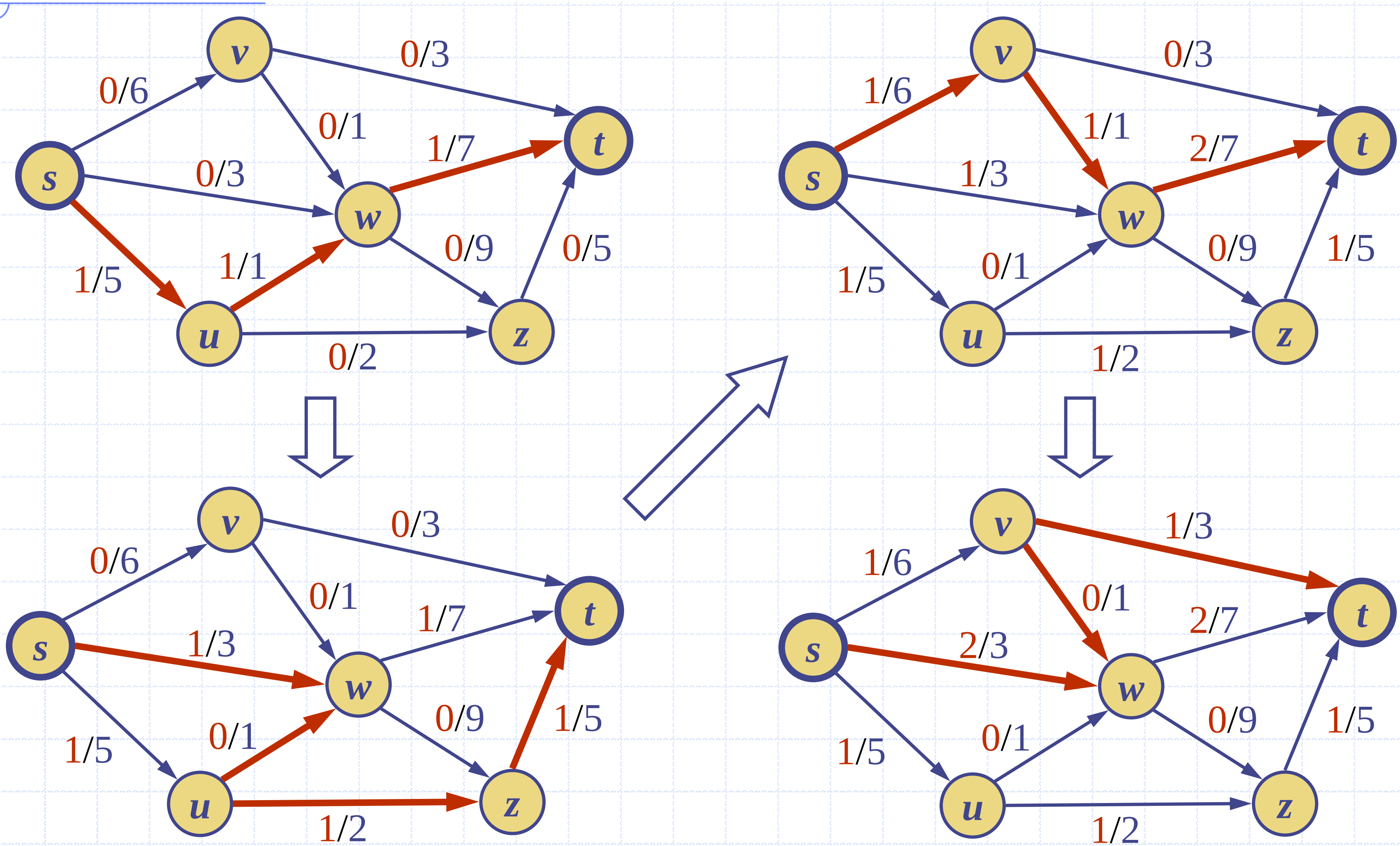
Theorem:

The value of a maximum flow is equal to the capacity of a minimum cut

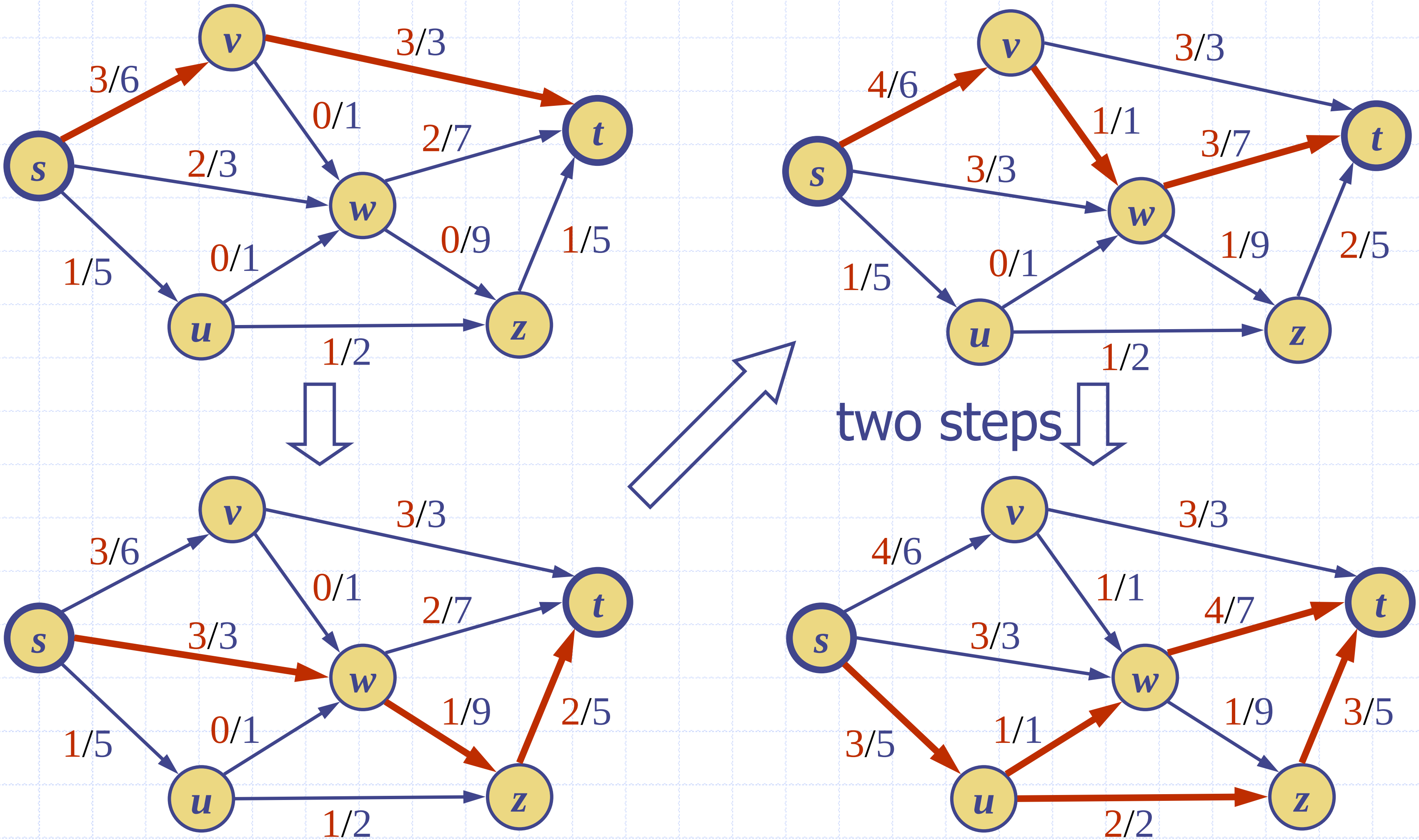


$c(\chi) = |f| = 10$

Example (1)

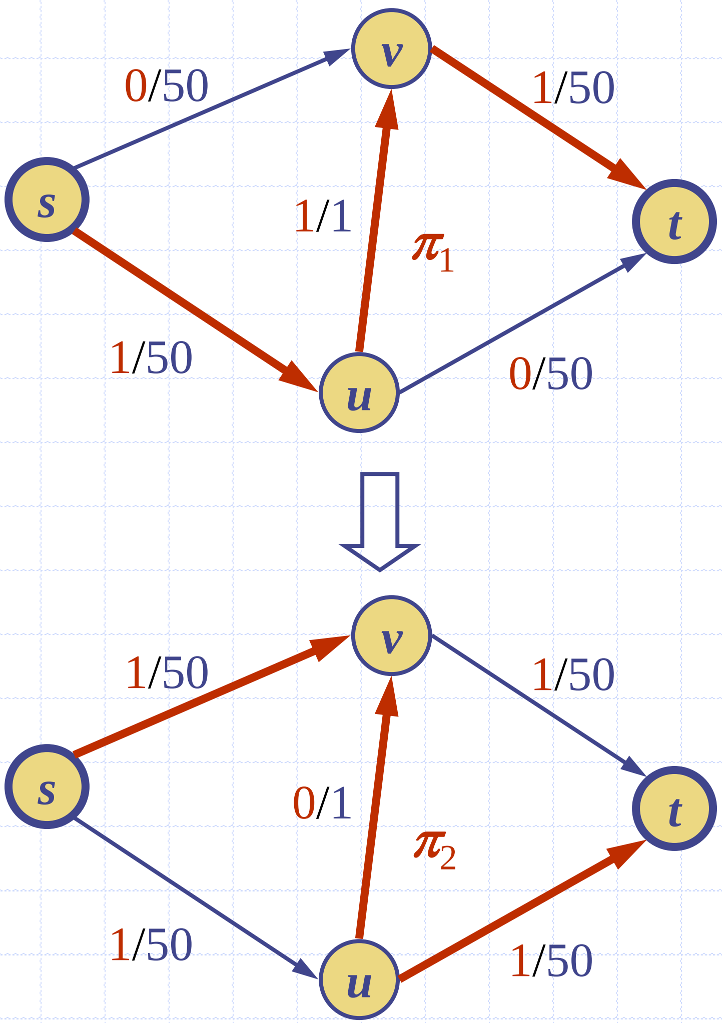

v
0/3
0/6
0/1
1/7
t
s
0/3
w
0/9
0/5
1/5
1/1
u
z
0/2
v
0/3
0/6
0/1
1/7
t
s
1/3
w
0/9
1/5
1/5
0/1
u
z
1/2
v
0/3
1/6
1/1
2/7
t
s
1/3
w
0/9
1/5
1/5
0/1
u
z
1/2
v
1/3
1/6
0/1
2/7
t
s
2/3
w
0/9
1/5
1/5
0/1
u
z
1/2

Example (2)


v
3/3
3/6
0/1
2/7
t
s
2/3
w
1/5
0/1
0/9
1/5
u
z
1/2
v
3/3
4/6
1/1
3/7
t
s
3/3
w
1/5
0/1
1/9
2/5
u
z
1/2
two steps
v
3/3
3/6
0/1
2/7
t
s
3/3
w
1/5
0/1
1/9
2/5
u
z
1/2
v
3/3
4/6
1/1
4/7
t
s
3/3
w
3/5
1/1
1/9
3/5
u
z
2/2

Analysis

- In the worst case, Ford-Fulkerson's algorithm performs $|f^*|$ flow augmentations, where f^* is a maximum flow
- Example
  - The augmenting paths found alternate between π_1 and π_2
  - The algorithm performs 100 augmentations
- Finding an augmenting path and augmenting the flow takes $O(n + m)$ time
- The running time of Ford-Fulkerson's algorithm is $O(|f^*|(n + m))$



Maximum Bipartite Matching

- In the maximum bipartite matching problem, we are given a connected undirected graph with the following properties:
  - The vertices of G are partitioned into two sets, X and Y.
  - Every edge of G has one endpoint in X and the other endpoint in Y.
- Such a graph is called a **bipartite graph**.
- A **matching** in G is a set of edges that have no endpoints in common—such a set "pairs" up vertices in X with vertices in Y so that each vertex has at most one "partner" in the other set.
- The maximum bipartite matching problem is to find a matching with the greatest number of edges.

Reduction to Max Flow

Let G be a bipartite graph whose vertices are partitioned into sets X and Y. We create a flow network H such that a maximum flow in H can be immediately converted into a maximum matching in G:

- We begin by including all the vertices of G in H, plus a new source vertex s and a new sink vertex t.
- Next, we add every edge of G to H, but direct each such edge so that it is oriented from the endpoint in X to the endpoint in Y. In addition, we insert a directed edge from s to each vertex in X, and a directed edge from each vertex in Y to t. Finally, we assign to each edge of H a capacity of 1.

- Given a flow f for H, we use f to define a set M of edges of G using the rule that an edge e is in M whenever f(e) = 1.

Example and Analysis

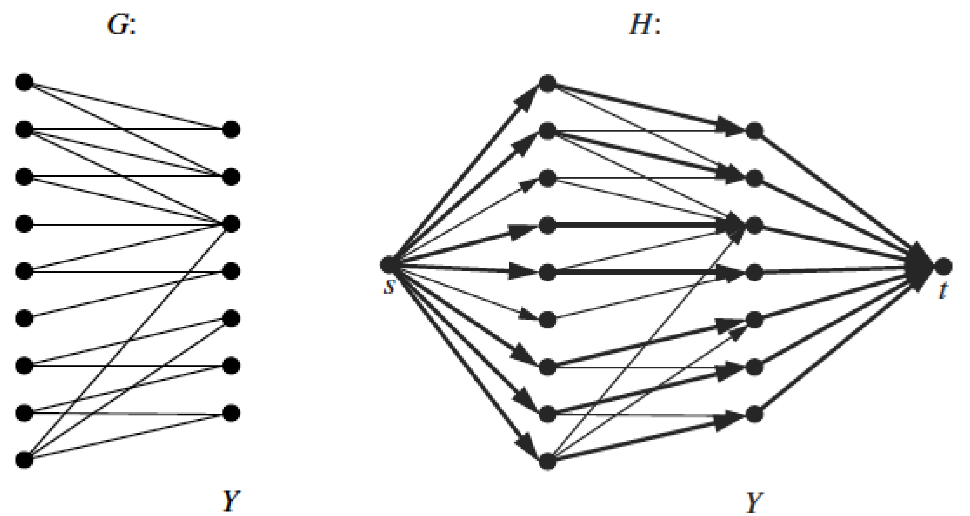


Figure 16.11: (a) A bipartite graph G. (b) Flow network H derived from G and a maximum flow in H; thick edges have unit flow and other edges have zero flow.

- Running time is O(nm), because G is connected.